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Wave action and set-down for waves on a shear current

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This paper considers steady, slowly varying water waves propagating over a gently sloping bed on a steady current. The current varies linearly with depth, and so has constant vorticity Ω. The analysis is two-dimensional and dissipation is neglected. Definitions, and expressions correct to second order in the amplitude, are given for the radiation stress, wave energy density E and total energy flux. An average Lagrangian $\mathscr{L}$, obtained by heuristic arguments from Clebsch potentials, leads to the result that for this particular problem E equals the wave action $\mathscr{L}_\omega$ times the angular frequency ω_{rm} relative to a frame of reference moving with the *average-over-depth* current velocity U_m. This determines the variation of the amplitude with distance explicitly. An analytical expression for the height of the mean water surface is found by a heuristic argument which compares the conservation equations for total energy and wave action. All the results have been checked directly by substitution back into the basic equations. Graphs illustrate the effect of the vorticity Ω on the wavelength, amplitude and set-down.

1. Introduction

This study concerns the propagation of surface gravity waves on a shear current over a slowly varying bed. It is restricted to two-dimensional flow without dissipation.

Waves on a rotational current have not been studied very extensively so far. Thompson (1949) introduced the idea of modelling a real current profile by a number of straight lines. He also showed how the dispersion relation may be found from the kinematic surface condition, a result which was found independently by Biesel (1950). Abdullah (1949) found the dispersion relation for waves on a current varying exponentially with depth. In a classical paper Burns (1953) studied long waves on an arbitrary current and found a simple integral expression for the dispersion relation. In an appendix to Burns' paper Lighthill discussed the critical Froude number above which no upstream propagation is possible. Results (including the effect of surface tension) were presented for a $\frac{1}{7}$ power law for the velocity distribution with depth. A more general discussion (abandoning the long-wave theory) of the influence of this current profile on the wavelength was given by Hunt (1955). Taylor (1955) studied

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two special bilinear profiles describing a surface current acting as a hydraulic breakwater in deep water, and Brevik (1976) extended this work to include finite depths. Sun' Tsao (1959) found expressions for the surface displacement and particle velocities in waves on a one-line current profile (allowing an amplitude-dependent mass flux; see below). Dalrymple (1973) presented the dispersion relation and the stream function for linear waves on an arbitrary bilinear current profile, and developed numerical procedures for the calculation of the surface elevation and velocity distribution for higher-order waves on linear and bilinear profiles and for waves on arbitrary currents. Dalrymple & Cox (1976) assumed that the vorticity varied linearly with the stream function and computed results showing the influence of the vorticity on the horizontal velocity profiles, wavelength and crest elevation. Fredsøe (1974) used a cosine velocity profile in his investigation of stationary waves.

The above-mentioned papers are based on very similar assumptions and a survey can be obtained from Fenton (1973).

Experiments studying current–wave interaction are rare. Yu (1952) and Sarpkaya (1955, 1957) were among the first to perform flume experiments of this type.

The aim of our study is to calculate the variations in the wave amplitude, the wavelength, the position of the mean water surface and the current velocity with depth when the mean volume flux q and the absolute wave frequency ω are both constant. Results are presented for a one-line ('linear') current profile, which is a good approximation to several flows important in coastal engineering practice.

It is assumed that the parameters which characterize the current–wave motion are constant in time. We shall further introduce a key assumption stating that, below the wave trough level, all contributions to the Eulerian-mean flow of second order in the wave amplitude are independent of depth (figure 6). This is appropriate to the phenomena in shoaling water which motivate this study.

New results include analytical expressions for the wave amplitude, the position of the mean water surface, the radiation stress S, the wave energy density E and the total mean energy flux F. The pioneering work of Longuet-Higgins & Stewart (1960, 1961, 1962, 1964) dealt with the corresponding expressions for irrotational flow. We shall see that our definitions of S and E are natural ones for problems where the above key assumption is appropriate; when they are introduced into the expression for the energy flux this takes exactly the same form as for irrotational flow. More surprising, perhaps, is the extremely simple connexion between E and Whitham's adiabatic invariant $\mathscr{L}_\omega$ as detailed below.

A special average Lagrangian $\mathscr{L}$ plays an important role in this study. This concept and some of its applications were presented in a series of papers by Whitham, beginning in 1965; see Whitham (1974).

Bretherton & Garrett (1968) showed that for linear waves and irrotational flow the wave energy, calculated in the frame of reference moving with the local basic flow, which is depth-independent in their case, could be interpreted as Whitham's adiabatic invariant times the intrinsic frequency, defined as the frequency relative to the same moving frame of reference. In our study a Lagrangian $\mathscr{L}$ for rotational flow is constructed and a non-trivial extension of the result for irrotational flow is found, namely that E simply equals $\mathscr{L}_\omega$ times the intrinsic frequency, where 'intrinsic' means 'relative to the special frame of reference $\mathscr{F}_m$ in which the average-over-depth velocity is brought to zero'. The uniqueness of $\mathscr{F}_m$ is further accentuated by the disappearance of the interaction term (containing the radiation stress) in the expression for the total energy flux with the mean water surface as the reference level, calculated in this special frame; see § 2.

Further studies on wave-action conservation for linear solutions in slowly varying wave guides were presented by Bretherton (1968), together with the first justification of Whitham's method in terms of formal asymptotic expansions. An interesting generalization of Whitham's adiabatic conservation law is given by Hayes (1970). He also includes an example somewhat similar to our problem, viz. non-dissipative wave propagation in an acoustic duct, in the presence of a basic flow whose velocity is a function of position in the cross-section. However, his integrated action density $\mathscr{A}$ is not generally 'interpretable as intrinsic energy density divided by intrinsic frequency' (Hayes 1970, p. 198). According to Bretherton (1978) and McIntyre (1977), Hayes' work also demonstrates a close connexion between wave-action conservation and the classical conservation laws for the 'energy–momentum tensor' (Landau & Lifshitz 1975).

The results discussed in the following sections contain three well-known special cases.

(i) For a pure current some trivial results emerge on putting $a = 0$, a being the wave amplitude, such as the expression for the height of the mean water surface, etc.

(ii) Stokes' second-order wave theory is obtained by putting $\Omega = 0$ and $U_m = 0$, Ω being the vorticity of the current profile and U_m being the average-over-depth velocity (which vanishes in the frame $\mathscr{F}_m$).

(iii) The theory of the irrotational combination of waves and a current over a slowly varying bed emerges by putting $\Omega = 0$. Jonsson, Skougaard & Wang (1971) solved this problem analytically and presented graphs showing the variation of the wave-length and height with water depth.

Jonsson (1978) discusses the extension to two horizontal dimensions for irrotational flow. Other topics which are not of direct interest here are described in surveys by Peregrine (1976) and Jonsson (1977).

An Eulerian description of the fluid motion is used throughout the present paper.

2. Radiation stress, wave energy density and total energy flux

It is convenient to define a 'formal current velocity profile'

$$U(z) = U_s + \Omega z,$$

which is the current profile that would exist in the absence of waves for a constant mean water depth. This is shown in figure 1. The vorticity Ω is a given constant (shown positive in the figure). U_s is a formal surface velocity which is related by definition to the mean volume flux q and the mean water depth h by

$$q = U_s h - \tfrac{1}{2}\Omega h^2. \tag{1}$$

We shall specify that q is exactly constant. A discussion of the relationship between the formal current profile and the real Eulerian-mean current profile is given in the appendix.

By the use of a regular perturbation technique with the wave steepness as indicator of smallness (analogous to Stokes' wave theory) a solution to arbitrary order can be

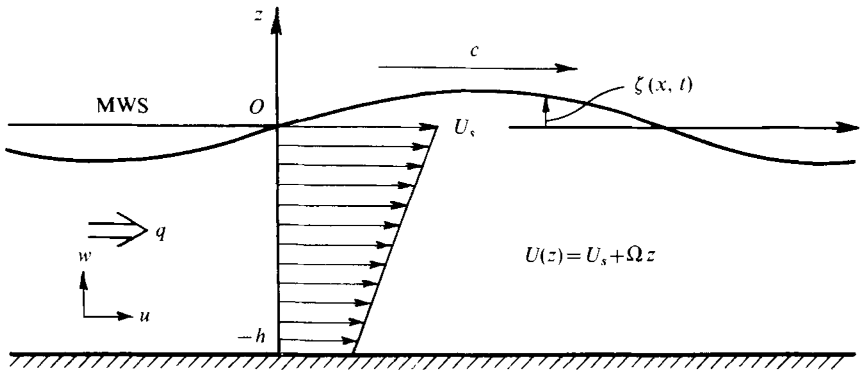

FIGURE 1. Definition sketch. Horizontal bed.

found for wave propagation on this shear flow. The problem is to solve the local momentum and mass conservation equations using the kinematic bottom and surface conditions, as well as the dynamic surface condition, and demanding a periodic solution. This problem was formulated by Sun' Tsao (1959), who found expressions for the surface displacement $\zeta(x,t)$ and total velocity components $u(x,z,t)$ and $w(x,z,t)$. In his treatment, the mean volume flux turns out to be a function of the wave amplitude; however, as already indicated, we find it more convenient to require *ab initio* that q is constant (to all orders). This slightly rearranges the x-independent terms in the expansion. The resulting formulae correct to second order for $\zeta(x,t)$, $u(x,z,t)$, $w(x,z,t)$ and the pressure $p(x,z,t)$ were given by Brink-Kjær & Jonsson (1975) and Brink-Kjær (1976). These are lengthy and will not be reproduced here. We adopt the convention that ζ is measured from the mean water surface, so that its expression contains no x-independent term. The wave amplitude a is therefore defined such that

$$\zeta = a\cos\theta + O(a^2), \tag{2}$$

where the $O(a^2)$ term is proportional to $\cos 2\theta$ and $\theta(x,t)$ is the phase function, with $\partial\theta/\partial t = \omega =$ absolute frequency and $-\partial\theta/\partial x = k =$ wavenumber $= 2\pi/$wavelength. The phase velocity c_{rs} relative to the formal surface velocity is

$$c_{rs} \equiv c - U_s, \tag{3}$$

where $c = \omega/k$ is the absolute phase velocity. The dispersion relation, originally presented by Thompson (1949) and Biesel (1950), then takes the form

$$c_{rs}^2 = (g - \Omega c_{rs})\, k^{-1} \tanh kh, \tag{4}$$

where g is the acceleration due to gravity.

The accuracy of (4) for real flows has been tested in the following way. In his theoretical and experimental study of stationary waves on a current, Fredsøe (1974) calculated the dispersion relation for a cosine current velocity profile

$$U = U_s \cos(1{\cdot}2z/h),$$

which was found to model flows over a rough bottom in the laboratory very well. His measurements (his figure 3*b*) are plotted in our figure 2, together with three theoretical dispersion relations for stationary waves, namely two with shear (the

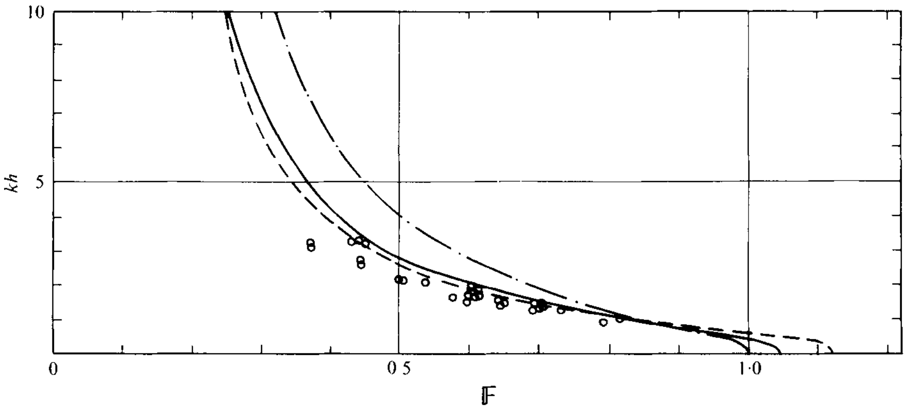

FIGURE 2. Dimensionless wavenumber kh *vs.* Froude number $\mathbb{F} = q/(g^{1/2}h^{3/2})$ for stationary waves over a rough bed. ○, measurements (Fredsøe 1974); ----, cosine current profile; ——, linear current profile; —·—, irrotational flow.

cosine and linear profiles, each having the same surface velocity U_s) and one corresponding to irrotational flow. The linear-current curve is found from (3) and (4) with $c = 0$. It is seen that the linear-profile approximation gives a good representation of the rather pronounced influence of vorticity on the wavelength, for a given volume flux q and depth h.

The absolute group velocity $c_g \equiv \partial\omega/\partial k$ is found from (4) to be

$$c_g = U_s + \frac{g(1+G) - \Omega c_{rs} G}{2g - \Omega c_{rs}} c_{rs}, \tag{5}$$

in which

$$G \equiv 2kh/\sinh 2kh. \tag{6}$$

In deep water $G = 0$ while in shallow water it approaches unity.

The mean water surface (MWS) is horizontal when waves and currents characterized by constant parameters propagate without dissipation over a horizontal bed. Thus the mean water surface is an obvious choice for a reference level in this case, so we take the origin of co-ordinates to be at that level. The following definitions (exact) and expressions (correct to second order) for the radiation stress S, the mean *wave* energy density E and the *total* mean energy flux F_{MWS} all have the mean water surface as a reference level (see Brink-Kjær & Jonsson 1975).

$$S \equiv \overline{\int_{-h}^{\zeta} p\, dz} - \int_{-h}^{0} (-\rho g z)\, dz + \rho \overline{\int_{-h}^{\zeta} u^2\, dz} - \rho \int_{-h}^{0} U^2 dz, \tag{7}$$

which gives (on substitution from the detailed solutions cited above)

$$S = \tfrac{1}{4}\rho g a^2(1+2G) + \tfrac{1}{4}\rho\Omega^2 a^2 h + \tfrac{1}{2}\rho\Omega c_{rs} a^2 kh \tanh kh, \tag{8}$$

where a bar denotes averaging over the (constant) absolute period (or wavelength) and ρ is the density.

$$E \equiv \overline{\int_{-h}^{\zeta} \rho g z\, dz} - \int_{-h}^{0} \rho g z\, dz + \tfrac{1}{2}\rho \overline{\int_{-h}^{\zeta} (u^2 + w^2)\, dz} - \tfrac{1}{2}\rho \int_{-h}^{0} U^2 dz, \tag{9}$$

which gives

$$E = \tfrac{1}{2}\rho g a^2 + \tfrac{1}{8}\rho\Omega^2 a^2 h + \tfrac{1}{4}\rho\Omega c_{rs} a^2(-1 + G + kh \tanh kh). \tag{10}$$

Half the first term is the excess potential energy; the remainder comes (after manipulations involving use of the dispersion relation) from the kinetic energy.

$$F_{\mathrm{MWS}} \equiv \overline{\int_{-h}^{\zeta} [p+\rho g z+\tfrac{1}{2}\rho(u^2+w^2)]\,u\,dz}, \tag{11}$$

which gives (after some lengthy manipulations)

$$F_{\mathrm{MWS}} = \tfrac{1}{2}\rho\int_{-h}^{0} U^3 dz + c_g E + S U_m,\dagger \tag{12}$$

where $U_m \equiv q/h = U_s - \frac{1}{2}\Omega h$, the average-over-depth velocity. Note that, in (7), (9) and (11), u is still the *total* horizontal particle velocity. As already mentioned, the expression for F_{MWS} is formally identical with the corresponding expression for irrotational flow ($U(z)$ = constant, i.e. $\Omega = 0$). In the latter case, (8), (10) and (12) naturally reduce to well-known expressions; see Longuet-Higgins & Stewart (1960) and Jonsson *et al.* (1971). In particular E is then the wave energy in the sense whose general meaning and physical interpretation were clarified by Bretherton & Garrett (1968).

If we choose another reference level for the potential energy, the limits of integration in (7) and (9) will be altered in such a way that the final expressions for S and E remain unchanged, i.e. S and E have been so defined as to be independent of the chosen reference level. They are also Galilean invariant. Neither statement is true for the total mean energy flux, however. For a reference level such that the mean water surface is at a height b above it, we find the total mean energy flux F_b to be

$$F_b = F_{\mathrm{MWS}} + \rho g h b U_m. \tag{13}$$

The last term stems from the new reference level for the potential energy. This expression is immensely important for flow over a non-horizontal bed, since here $b = b(x)$; see the next section.

The reference frame $\mathscr{F}_m$ (in which the average-over-depth velocity is zero) obviously has a special significance in that the mean volume flux vanishes. *In that frame*, furthermore, the interaction term containing the radiation stress disappears from the expression for the energy flux, as can be seen below. In a frame moving in the x direction with an arbitrary velocity Q, the following expression emerges for the total mean energy flux, referred to the mean water surface:

$$F^{Q}_{\mathrm{MWS}} = \tfrac{1}{2}\rho\int_{-h}^{0} (U-Q)^3 dz + (c_g - Q)E + S(U_m - Q),$$

in which U, c_g and U_m are still velocities in the fixed frame. Thus *only* in $\mathscr{F}_m$ does the interaction term (the last term) disappear, since here $Q = U_m$.

† Integrals over the mean depth of the current velocity squared and cubed appear in a number of expressions. They read

$$\int_{-h}^{0} U^2 dz = U_m^2 h + \tfrac{1}{12}\Omega^2 h^3, \quad \int_{-h}^{0} U^3 dz = U_m^3 h + \tfrac{1}{4} U_m \Omega^2 h^3.$$

For clarity and identification the integral definitions are retained in the above expressions, however.

3. A slowly varying bed

We shall now study the spatial dependence of wave parameters for waves propagating on a steady shear current over a slowly varying sea bed. Thus the dynamical problem for the waves is independent of time. We further restrict attention to solutions characterized by a single, constant absolute frequency ω.

When the mean flow is not horizontal, the mean water surface no longer forms a constant level. This is sketched in figure 3, where $b = b(x)$ is the height of the mean water surface. (The 'current–wave set-down' $b_1 - b$, where b_1 is some constant reference height, involves effects due to the waves, to the Bernoulli effect in the basic current and to interactions.) In this case there appears to be no natural reference level for the potential energy and the position of mean water surface, so the x axis is placed at an arbitrary, but fixed horizontal level.† It is assumed that we have a slowly varying mean state such that *locally* the horizontal-bed expressions for dispersion, wave energy density, etc., are still valid. This essentially corresponds to the classical WKBJ approximation. Since dissipation will be neglected, the vorticity remains unchanged. This, combined with the assumption of a slowly varying depth, shows that the assumption that the vorticity is constant with depth at just one position leads to a linear velocity profile with the same constant gradient in the vertical direction everywhere. So in addition to q and ω, Ω can also be assumed constant.

The motion of waves riding on a current over a non-horizontal bed is then characterized by the slowly varying quantities U_s, k, a, h and the position b of the mean water surface; see figure 3. These variables will depend on x, but not on time t since in this study we are concerned with steady waves and currents.

The formal surface velocity U_s and wavelength $2\pi/k$ are known functions of the mean water depth; see (1) and (4). The wave amplitude a, the height of the mean water surface b and hence the mean water depth h can be found from equations expressing conservation of total momentum and total energy.

The mean *total* momentum flux M over a depth h (per unit wave front) equals the sum of the first and third terms on the right-hand side of (7), so we find

$$M = S + \rho \int_{-h}^{0} U^2 dz + \tfrac{1}{2}\rho g h^2. \tag{14}$$

The horizontal component of the mean total pressure force acting on the fluid at the bed per unit length in the x direction is

$$P_h = \rho g h \, dD/dx. \tag{15}$$

Here $-dD/dx$ is the bed slope, since $D = D(x)$ is the (known) position of the bed;

† This is in contrast to irrotational and time-periodic free-surface flow. For that case Jonsson *et al.* (1971) define a horizontal level, the so-called mean energy level, which they show to be connected with the flow in a unique and natural way. If just one 'streamline' goes from infinite to finite depth, this level coincides with the mean water surface in deep water, where the current speed vanishes. It has also been shown by the senior author (Jonsson 1971) that, if one uses this mean energy level as a reference level for the potential energy, the equation for conservation of the total energy is exactly equivalent to that of wave-action conservation. Extending this to two horizontal dimensions (Jonsson 1978) shows that the wave *ray* concept arises in a natural way, since here the total energy equation transforms to one of wave-action conservation between rays.

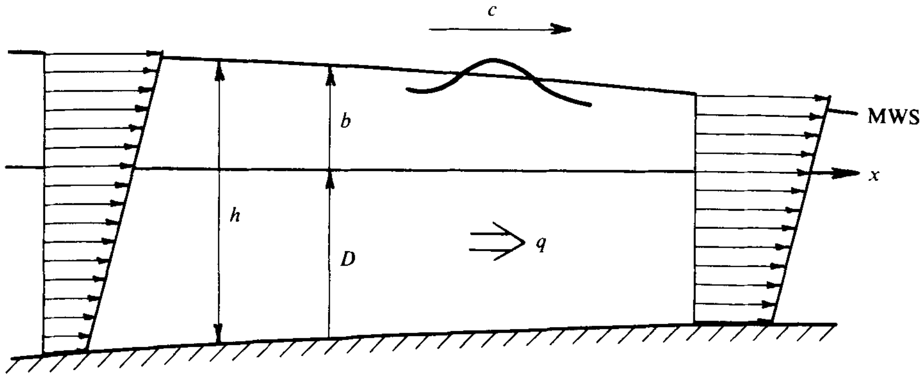

FIGURE 3. Definition sketch. Slowly varying bed.

see figure 3. The equation $-dM/dx+P_h = 0$ of total momentum conservation therefore takes the form

$$\frac{dS}{dx}+\frac{d}{dx}\left(\rho\int_{-h}^{0}U^2dz\right)+\rho gh\frac{db}{dx} = 0, \tag{16}$$

since $h = D+b$ from figure 3.

The equation expressing total energy conservation is simply $dF_b/dx = 0$, where F_b is given by (13). From (12), this becomes

$$\frac{d}{dx}\left(\tfrac{1}{2}\rho\int_{-h}^{0}U^3dz+c_gE+SU_m+\rho ghbU_m\right) = 0. \tag{17}$$

Using again the fact that $h = D+b$, it is readily seen that a and b can be found as functions of h instead of x. This is used in figures 4 and 5.

The mean water surface height can be eliminated from (16) and (17) to give

$$\frac{d}{dx}(c_gE)+S\frac{dU_m}{dx}= 0. \tag{18}$$

(The need for the 'extra' term SdU_m/dx was pointed out by Longuet-Higgins & Stewart (1961) for the case of irrotational flow. Wave energy, as defined here, is not a conserved quantity.) The wave amplitude a is determined uniquely by (18), and this equation could be used directly for numerical computation. But it will be shown subsequently that in fact an analytical solution to (18) exists; this solution is not obvious by inspection or manipulation of (18), but can be arrived at very efficiently by means of the heuristic argument which follows in the next two sections. The results will then be checked rigorously by direct substitution back into (18).

4. An average Lagrangian

Another method for dealing with a slowly varying wave train is to use an average Lagrangian; see Whitham (1974, p. 393 ff.). For rotational flow Luke (1967) proposed that the motion can be described by a Lagrangian density (per unit x distance)

$$L = -\int_{-h(x)}^{\zeta(x,t)}\rho[\phi_t+\alpha\beta_t+\tfrac{1}{2}\mathbf{u}^2+gz]\,dz, \tag{19}$$

where the velocity field $\mathbf{u}$ is described by Clebsch potentials $\phi(x,z,t)$, $\alpha(x,z,t)$ and $\beta(x,z,t)$, i.e.

$$\mathbf{u} = \nabla\phi + \alpha\nabla\beta. \tag{20}$$

(The Clebsch representation (20) may not allow completely general motion if ϕ, α and β are to be single valued (Bretherton 1970, § 6), but this need not deter us from using it for the present, heuristic purpose.)

Assuming as before a solution periodic in the phase function $\theta(x,t)$, we can define an average Lagrangian as

$$\mathscr{L} \equiv \frac{1}{2\pi}\int_0^{2\pi} L\,d\theta \tag{21}$$

and, by using the known structure (§ 2) for periodic waves, express it as a function of $\omega = \partial\theta/\partial t$, $k = -\partial\theta/\partial x$ and a (and also of course g, ρ, h, Ω and U_s). We then apply Whitham's principle that the variational derivatives of $\mathscr{L}$ with respect to a and θ should vanish, yielding respectively

$$\mathscr{L}_a = 0 \tag{22}$$

and

$$\partial\mathscr{L}_\omega/\partial t - \partial\mathscr{L}_k/\partial x = 0, \tag{23}$$

where the subscripts denote partial differentiation of the known functional form $\mathscr{L}(\omega,k,a)$.

It is not easy to calculate $\mathscr{L}$ directly from (19). Instead, we argue from the facts (Whitham 1974, p. 393) that (22) must be a form of the dispersion relation and that, correct to $O(a^2)$, $\mathscr{L}$ must be of the form

$$\mathscr{L} = a^2G(\omega,k;g,\ldots)+\mathscr{L}_0(g,\ldots). \tag{24}$$

Here $\mathscr{L}_0$ does not explicitly include a, ω or k, and so will not contribute to (22) and (23); see also Whitham (1974, p. 393). In (24), $G(\omega,k) = 0$ must be a form of the dispersion relation.

Suppose that the known form of the dispersion relation (4) is written as

$$F(\omega,k;g,\ldots) \equiv kc_{rs}^2/\tanh kh + \Omega c_{rs} - g = 0. \tag{25}$$

Recalling (22), it then appears that (24) must take the form

$$\mathscr{L} = a^2 f(\omega,k;g,\ldots)\,F(\omega,k;g,\ldots)+\mathscr{L}_0(g,\ldots), \tag{26}$$

where $f \neq 0$. Now the $O(a^2)$ contribution to (19), which contains the factor g and arises directly from the linearized surface displacement (2), is easily seen to be $-\frac{1}{4}\rho g a^2$. The simplest possibility is that this contribution corresponds to the last term in (25), so that

$$f = \tfrac{1}{4}\rho. \tag{27}$$

Then (25) and (27) indicate that

$$\mathscr{L} = \tfrac{1}{4}\rho g a^2\left(\frac{(\omega-kU_s)^2}{gk\tanh kh}+\frac{\Omega(\omega-kU_s)}{gk}-1\right)+\mathscr{L}_0, \tag{28}$$

since from (3) we have $c_{rs} = (\omega-kU_s)/k$.

An immediate partial check on the correctness of (28) is that it reduces when $\Omega = 0$ to the well-known result for irrotational flow (see, for example, Whitham 1974, p. 555). Equation (23) is Whitham's adiabatic conservation relation for unsteady two-dimensional flow. Here we have made the self-consistent assumptions of a time-independent medium and steady waves, so (23) becomes

$$d\mathscr{L}_k/dx = 0. \tag{29}$$

Since we have already calculated c_g, $\mathscr{L}_k$ is most easily found from the relation

$$\mathscr{L}_k = \mathscr{L}_\omega(\partial\omega/\partial k)_{x=\text{constant}} = \mathscr{L}_\omega c_g, \tag{30}$$

which holds when (22) and (24) hold. So, calculating $\mathscr{L}_\omega$ from (28) and introducing the group velocity through (5), we end up with

$$\mathscr{L}_k = \tfrac{1}{4}\rho a^2\left(\frac{2c_{rs}}{\tanh kh}+\frac{\Omega}{k}\right)\left(U_s+\frac{g(1+G)-\Omega c_{rs}G}{2g-\Omega c_{rs}}\,c_{rs}\right). \tag{31}$$

It may be shown directly, without relying on (28), that (29) and (31) are equivalent to (18); the details are straightforward but tedious, and are omitted. This is taken as justifying the choice (27), and hence the tentative result (28).

5. Wave-action conservation

The expression (31) admits further simplification. From Bretherton & Garrett (1968) we know that for linear waves on an irrotational current Whitham's adiabatic invariant $\mathscr{L}_\omega$ can be written in a form analogous to the classical adiabatic invariant, i.e.

$$\mathscr{L}_\omega = E/\omega_r, \tag{32}$$

in which $\omega_r = \omega - kU$ is the intrinsic angular frequency, or the frequency in a frame of reference moving with the local mean flow velocity U. Bretherton & Garrett called this form of $\mathscr{L}_\omega$ the 'wave action'. Here the situation is more complex since ω_r is a function of z; in fact, it is not altogether obvious that a physically well-defined frequency exists in terms of which $\mathscr{L}_\omega$ can be written in a form analogous to (32). However, if we introduce the angular frequency ω_{rm} relative to the special frame of reference $\mathscr{F}_m$ encountered previously, which moves with the *average-over-depth* velocity U_m, i.e.

$$\omega_{rm} = \omega - kU_m, \tag{33}$$

then the following simple relation emerges from (30) and (31):

$$\mathscr{L}_\omega = E/\omega_{rm}. \tag{34}$$

Here E is just the wave energy density (10) which follows from the definition (9). This also means that E can be written more concisely as

$$E = \tfrac{1}{4}\rho a^2(2g-\Omega c_{rs})\,\omega_{rm}/\omega_{rs}, \tag{35}$$

where $\omega_{rs} = \omega - kU_s$ is the frequency relative to the surface velocity.

Thus we have shown that for steady waves on a linear shear current $E = \omega_{rm}\mathscr{L}_\omega$. This result could hardly have been guessed from the results in §§ 2 and **3**.

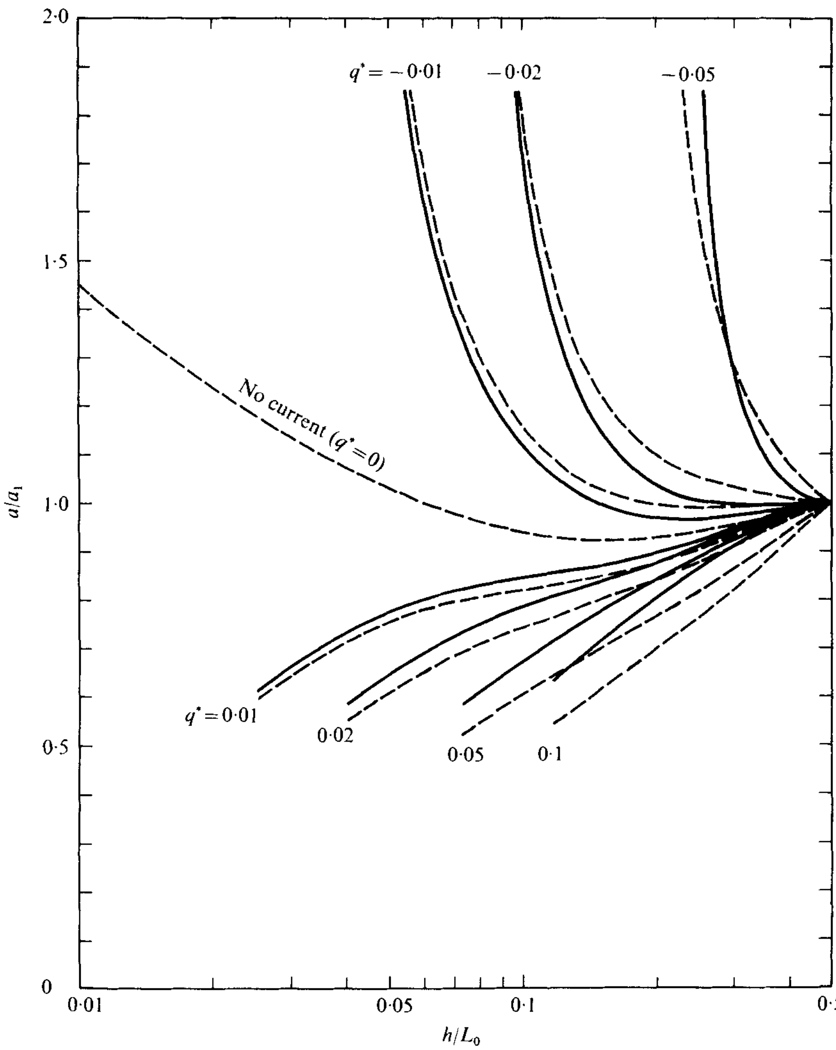

FIGURE 4. Relative amplitude a/a_1 *vs.* dimensionless depth h/L_0 ($h_1/L_0 = 0{\cdot}5$). ——, $\Omega^* = 2q^*/(h_1/L_0)^2$; – – – –, $\Omega^* = 0$ (irrotational flow). For q^* positive, curves stop at a Froude number $\mathbb{F} = q/(g^{1/2}h^{3/2})$ equal to 1.

The wave amplitude variation can now be found by introducing (34) and (30) into (29):

$$\frac{d}{dx}\left(\frac{E}{\omega_{rm}}c_g\right) = 0, \tag{36}$$

in which the quantity in parentheses is the condensed version of (31). By analogy with the terminology introduced by Bretherton & Garrett, we adopt the term 'wave action' for E/ω_{rm}, the form taken by Whitham's invariant $\mathscr{L}_\omega$ in this problem. So we say that (36) expresses 'conservation of wave action', correct to $O(a^2)$.

For $\Omega = 0$ our result (36) reduces to the known result for irrotational flow, since our definition of E then coincides with that of Bretherton & Garrett. But it shows that an expression of exactly the same *form* as theirs applies to the more general case of a shear flow with constant vorticity Ω, provided that we make an appropriate choice for the associated local frame of reference.†

Figure 4 shows the variation of amplitude with water depth on a dimensionless plot. The integration constant in (36) is determined by the known (chosen) amplitude a_1 at some reference value of x where the water depth $h = h_1 = 0{\cdot}5L_0$ and by the known constant values of the discharge q, absolute frequency ω and vorticity Ω. In the figure $L_0 = 2\pi g/\omega^2$ is the linear, irrotational, no-current deep-water wavelength;

$$q^* = 2\pi q/(L_0^2\,\omega)$$

is a dimensionless discharge. For each value of q^* (except $q^* = 0$), two values of the dimensionless vorticity $\Omega^* \equiv 2\pi\Omega/\omega$ have been investigated, namely the irrotational case $\Omega^* = 0$ (dashed curves) and the case where the current velocity at the bed is zero at the reference depth h_1 (solid curves). The latter case corresponds to

$$\Omega^* = 2q^*/(h_1/L_0)^2.$$

The two sets of curves determine the outer limits for a realistic problem.

6. Position of mean water surface ('set-down')

The order of accuracy of our expression (28) for the average Lagrangian is too low to allow a determination of the height b of the mean water surface (figure 3). However, since (17) and (36) are both energy transport equations in conservation form, it is plausible that the quantities in parentheses in these equations are equal, apart from a constant factor and an arbitrary constant, and this suggests an expression for b. From similar considerations for irrotational flow (Jonsson 1971, 1978) it can be guessed that the 'missing' factor in (36) is ω, so

$$\tfrac{1}{2}\rho\int_{-h}^{0} U^3 dz + c_g E + SU_m + \rho g h b U_m = \frac{\omega}{\omega_{rm}} E c_g + \text{constant}, \tag{37}$$

which after some manipulation leads to

$$b = \underset{\text{I}}{-\frac{1}{q}\int_{-h}^{0}\frac{U^3}{2g}dz} \underset{\text{II}}{-\frac{a^2G}{4h}} + \underset{\text{III}}{\frac{a^2U_m}{2hc_{rs}}} \underset{\text{IV}}{-\frac{a^2\Omega}{8gh}(2U_m+\Omega h+2c_{rs}kh\tanh kh)} + \text{constant}, \tag{38}$$

where G is given by (6). It has been verified directly that this expression for b does satisfy (16). It is correct to $O(a^2)$. The constant is determined from the initial conditions.

The distance b is positive upwards; the 'set-down' could therefore be defined as $b_1 - b$, where b_1 is an initial value. With this interpretation (38) can be shown to lead to a number of well-known special results:

(i) A pure current ($a = 0$) leaves term I, which is the conventional hydraulic velocity head, or stagnation level.

† A referee has pointed out that there is another choice of reference frame which is natural from a different viewpoint, namely the frame in which the surface velocity equals zero. Redefining the wave energy as E' by calculating the kinetic energy solely from the perturbation (first-order) particle velocities, he finds $E' = \frac{1}{4}\rho a^2(2g - \Omega c_{rs})$. It is then seen from (34) and (35) that $\mathscr{L}_\omega$ can also be written as E'/ω_{rs}. However, in the authors' opinion this formalism is less attractive, primarily because the physical interpretation of E' is not obvious.

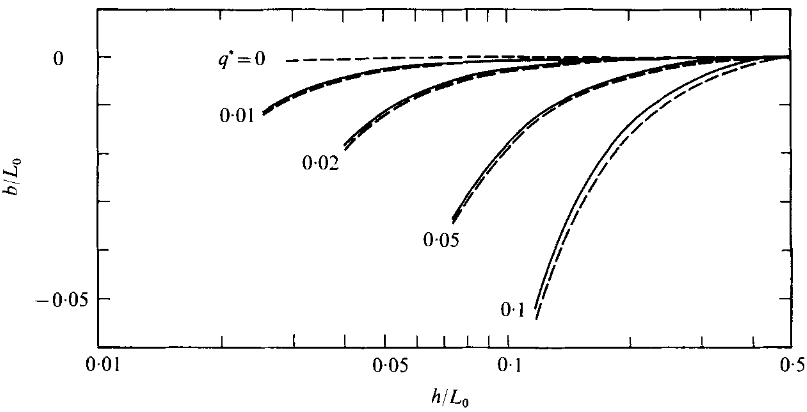

FIGURE 5. Dimensionless height of mean water surface b/L_0 *vs.* dimensionless depth h/L_0 ($h_1/L_0 =$ 0·5, $a_1/L_0 = 0{\cdot}01$, $b_1/L_0 = 0$). ——, $\Omega^* = 2q^*/(h_1/L_0)^2$; ----, $\Omega^* = 0$ (irrotational flow). For q^* positive, curves stop as in figure 4. For $q^* = 0$, the curve stops at $a/h = 0{\cdot}4$.

(ii) Pure waves in irrotational flow ($U_m = 0$, $\Omega = 0$) leave term II, which is the conventional wave set-down (Longuet-Higgins & Stewart 1962; Lundgren 1963; Phillips 1966, equation 3.7.6†).

(iii) Waves on a current in irrotational flow ($\Omega = 0$) leave terms I, II and III, which constitute the conventional current wave set-down (Jonsson *et al.* 1971).

Note that term III is a current–wave interaction term in the sense that it disappears in the two limits of vanishing waves and vanishing mean flow. So it cannot be easily checked. A rather simple way to find it, though, was presented by Jonsson (1971, 1978); see also Jonsson *et al.* (1971, equation (4.8)). Term IV is a vorticity–wave interaction term in the sense that it disappears for vanishing vorticity and vanishing waves. Generally it does not disappear for vanishing depth-averaged mean flow.

The variation of the mean water surface is shown in figure 5 (for positive currents only). L_0 and q^* are as defined in connexion with figure 4 and the two sets of curves correspond to those in the same figure. In contrast to the relative amplitude, the relative set-down is a function of the initial amplitude, so a value of a_1/L_0 must be specified as well. As would be expected, the effect of vorticity on the set-down is less dramatic than that on the amplitude (figure 4).

7. Conclusion

Periodic surface gravity waves propagating on a linear shear current over a sea bed of gentle slope in a time-independent and non-dissipative medium have been investigated for two-dimensional flow. The analysis was carried through by a combination of two methods: a classical perturbation technique leading to conservation equations for momentum and total energy, and a heuristic method using an incomplete average Lagrangian derived from the dispersion relation.

We have shown that, in the particular case of a linear current Whitham's adiabatic

† It is not obvious that Phillips' more complicated expression equals term II; noting that his $d(\coth\xi)/d\eta$ is $-g/[\omega c \sinh^2 kh(1+G)]$ and that his F is $\frac{1}{2}Ec(1+G)$ leads to this result.

invariant can be equated to the ratio between a wave energy density and an intrinsic angular frequency ω_{rm}, which arises in a natural way in our problem. The corresponding local frame of reference $\mathscr{F}_m$ is that in which the average-over-depth current velocity vanishes. This is a non-trivial extension of the analogous result for irrotational flow; it depends crucially on the assumption that the second-order contribution to the Eulerian-mean flow is independent of depth, below the wave trough level. It is not known at present whether there is a comparably simple general recipe for singling out such an intrinsic frequency in all kinds of wave problems.

An analytical expression for the height of the mean water surface was found by the heuristic device of comparing the conservation equations for total energy and wave action. It was verified by substitution back into the basic equations of §§ 2 and 3.

Graphs have been presented which illustrate the effect of vorticity on the wavelength, amplitude and set-down.

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Appendix. Discussion of the mean current profile

The mean volume flux q is by definition connected with the formal current through (1); see figure 1. However, between the trough and crest level there is a wave-induced net transport of fluid in the direction of wave propagation. This transport is balanced by a constant negative term in the expression for the total horizontal particle velocity u; see Brink-Kjær (1976). (This results from our convention of prescribing q in advance.) Below the wave trough level the Eulerian-mean particle velocity is, correct to second order,

$$\langle u \rangle = U_s + \Omega z - \frac{2g - \Omega c_{rs}}{4hc_{rs}} a^2. \tag{A 1}$$

Above the wave trough level $\langle u \rangle$ varies continuously with not more than one maximum (depending on the direction and strength of the current) and vanishes at the wave crest level (see figure 6 and Brink-Kjær 1976). It is dominated by the zero-order contribution associated with U_s; the next contribution is $O(a)$.

We have, by definition,

$$q = \overline{\int_{-h}^{\zeta} u\, dz} = \int_{-h}^{a} \langle u \rangle\, dz = \int_{-h}^{0} U(z)\, dz. \tag{A 2}$$

Figure 6 illustrates the difference between the Eulerian-mean current profile as defined here and the formal current profile. In the figure the current flows in the direction of wave travel. The average-over-depth velocity U_m is naturally the same for the two profiles in the figure, as is Ω.

In spite of this the definitions of S and E [see (7) and (9)] use the formal current profile, although it leads to expressions [see (8) and (10)] which are not bounded for increasing values of h, other parameters being constant. However, the simple results for F_{MWS} and wave-action conservation [see (12) and (36)] constitute our reasons for

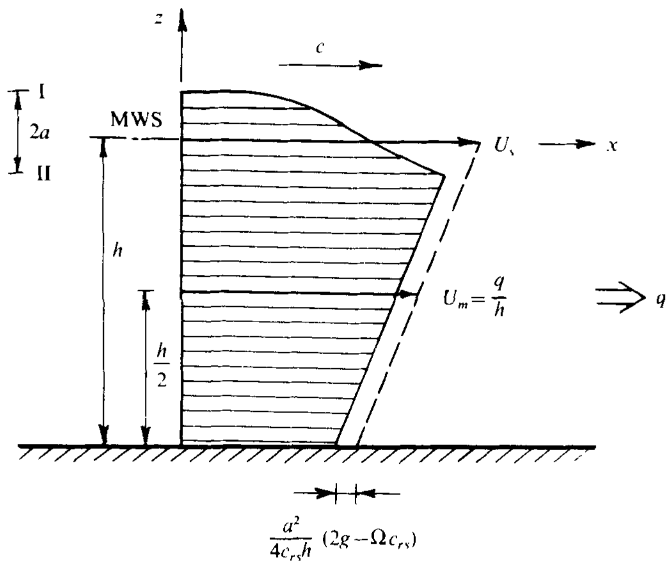

FIGURE 6. Mean velocity profiles for a positive current (schematic). ——, real (Eulerian) current profile; ----, formal current profile; I, wave crest level; II, wave trough level.

retaining these definitions. It should further be noted that, by virtue of (35), the last term in (A 1) can be written as $-E/(\rho h c_{rm})$, where $c_{rm} = c - U_m$, the phase velocity in the frame $\mathscr{F}_m$. This expression is formally the same as for irrotational flow. (Pursuing the philosophy presented in the footnote in § 5, the last term can alternatively be written as $-E'/(\rho h c_{rs})$.)

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